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2009 J. Phys. A: Math. Theor. 42 125201

(http://iopscience.iop.org/1751-8121/42/12/125201)

J. Phys. A: Math. Theor. **42** (2009) 125201 (9pp) doi:10.1088/1751-8113/42/12/125201

On the geometry of the first and second Painlevé equations*

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Received 10 July 2008, in final form 29 January 2009
Published 24 February 2009
Online at stacks.iop.org/JPhysA/42/125201

Abstract

In this paper we *explicitly* compute the transformation that maps the generic second-order differential equation $y'' = f(x, y, y')$ to the Painlevé first equation $y'' = 6y^2 + x$ (resp. the Painlevé second equation $y'' = 2y^3 + yx + \alpha$). This change of coordinates, which is a function of f and its partial derivatives, does not exist for every f; it is necessary that the function f satisfies certain conditions that define the equivalence class of the considered Painlevé equation. In this work we will not consider these conditions and the existence issue is solved *on line* as follows: if the input equation is known then it suffices to specialize the change of coordinates on this equation and test by simple substitution if the equivalence holds. The other innovation of this work lies in the exploitation of discrete symmetries for solving the equivalence problem.

PACS number: 02.30.Jr

1. Introduction

By fiber-preserving transformations we mean analytical transformations of the form

$$\mathbb{C}^2 \ni (x, y) \to (\bar{x}(x), \bar{y}(x, y)) \in \mathbb{C}^2$$

with the condition $\bar{x}_x \bar{y}_y \neq 0$ expressing their local invertibility. These transformations form a Lie pseudo-group with

$$\bar{x}_y = 0, \qquad \bar{x}_x \bar{y}_y \neq 0 \tag{1.1}$$

as a defining system.

* The research was supported in part by the Czech Ministry of Education, Youth and Sports within the project LC06002.

As indicated in the abstract, our aim is to *explicitly* compute the transformation of this form that maps the second-order equation

$$\mathcal{E}_f : y'' = f(x, y, y'),$$

where $y' = \frac{\mathrm{d}}{\mathrm{d}x} y(x)$, to the first Painlevé equation (resp. to the second Painlevé equation). This change of coordinates, which is clearly a function of f and its partial derivatives, does not exist for every f; it is necessary that the function f satisfies certain conditions that define the equivalence class of the considered Painlevé equation. Comparing to [KLS85] and [HD02], the existence issue is solved here *on line* as follows: if the input equation is known then it suffices to specialize the change of coordinates on this equation and test by simple substitution if the equivalence holds.

The calculations of transformation *candidates* are based on the following result [DP07]. Given a Lie pseudo-group of transformations Φ and denote by $\mathcal{S}_{\mathcal{E}_f,\Phi}$ the symmetry pseudo-group of the equation $\mathcal{E}_f$ w.r.t. Φ i.e., $\mathcal{S}_{\mathcal{E}_f,\Phi} = \Phi \cap \mathrm{Diff}^{\mathrm{loc}}(\mathcal{E}_f)$. In [DP07], we proved the following.

(i) The number of constants appearing in the change of coordinates is exactly the dimension of $\mathcal{S}_{\mathcal{E}_{\bar{f}},\Phi}$. Also, we have $\dim(\mathcal{S}_{\mathcal{E}_f,\Phi}) = \dim(\mathcal{S}_{\mathcal{E}_{\bar{f}},\Phi})$.

(ii) In the particular case when $\dim(\mathcal{S}_{\mathcal{E}_{\bar{f}},\Phi}) = 0$, the transformation φ is algebraic in f and its partial derivatives and it is obtained *without* solving differential equations. The degree of this transformation φ is exactly equal to the finite value card $(\mathcal{S}_{\mathcal{E}_{\bar{f}},\Phi})$.

The last case is exactly what happens when $\mathcal{E}_{\bar{f}}$ is one of the Painlevé equations and Φ is the pseudo-group of fiber-preserving transformations or more generally point transformations. Indeed, the classical Lie analysis shows that the point symmetry pseudo-group of each one of the Painlevé equations is zero dimensional. Moreover, according to the fact that the unique transformations that preserve the singularity structure are homographic transformations, one can show by straightforward computations that the point symmetry pseudo-group of Painlevé one is

$$\begin{cases} \bar{x} = x\dfrac{\bar{y}^2}{y^2}, \\ \bar{y}^5 = y^5, \end{cases} \tag{1.2}$$

and

$$\begin{cases} \bar{x} = x\dfrac{\bar{y}^2}{y^2}, \\ \bar{y}^3 = \dfrac{\bar{\alpha}}{\alpha} y^3, \\ \bar{\alpha}^2 = \alpha^2, \end{cases} \tag{1.3}$$

for Painlevé two when $\alpha \neq 0$ and

$$\begin{cases} \bar{x} = x\dfrac{\bar{y}^2}{y^2}, \\ \bar{y}^6 = y^6, \end{cases} \tag{1.4}$$

when $\alpha = 0$.

Fiber-preserving transformations are suitable when dealing with Painlevé equations. In particular, such transformations preserve the integrability in the sense of Poincaré [CM08]. However, since Painlevé equations lie in the class of equations of the form

$$y'' = A(x, y) + B(x, y)y' + C(x, y)y'^2 + D(x, y)y'^3$$

which is invariant under point transformations[1], we consider in the last section of this paper the equivalence under these more general transformations.

2. Building the invariants

Let $(x, y, p = y')$ be a system of local coordinates of $\mathrm{J}^1 = \mathrm{J}^1(\mathbb{C}, \mathbb{C})$, the space of first-order jets of functions $\mathbb{C} \ni x \to y(x) \in \mathbb{C}$ [Olv93]. Two scalar second-order ordinary equations

$$\mathcal{E}_f : y'' = f(x, y, y') \qquad \text{and} \qquad \mathcal{E}_{\bar{f}} : \bar{y}'' = \bar{f}(\bar{x}, \bar{y}, \bar{y}')$$

are said to be equivalent under a point transformation φ if its first prolongation (to J^1) maps the contact forms

$$\begin{cases} \omega^1 = \mathrm{d}y - p\,\mathrm{d}x \\ \omega^2 = \mathrm{d}p - f(x, y, p)\,\mathrm{d}x \end{cases}$$

to the contact forms

$$\begin{cases} \bar{\omega}^1 = \mathrm{d}\bar{y} - \bar{p}\bar{\mathrm{d}}x \\ \bar{\omega}^2 = \mathrm{d}\bar{p} - \bar{f}(\bar{x}, \bar{y}, \bar{p})\,\mathrm{d}\bar{x} \end{cases}$$

up to an invertible 2×2-matrix of the form

$$\begin{pmatrix} a_1 & 0 \\ a_2 & a_3 \end{pmatrix}.$$

The a_i are functions from J^1 to $\mathbb{C}$. To encode equivalence under fiber-preserving transformations (i.e., taking into account the Lie equations (1.1)) we must have

$$\varphi^* \,\mathrm{d}\bar{x} = a_4\,\mathrm{d}x$$

for a certain function $a_4 : \mathrm{J}^1 \to \mathbb{C}$. Summarizing, two second-order differential equations $\mathcal{E}_f$ and $\mathcal{E}_{\bar{f}}$ are equivalent under a fiber-preserving transformation φ if and only if

$$\varphi^* \begin{pmatrix} \mathrm{d}\bar{y} - \bar{p}\,\mathrm{d}\bar{x} \\ \mathrm{d}\bar{p} - \bar{f}(\bar{x}, \bar{y}, \bar{p})\,\mathrm{d}\bar{x} \\ \mathrm{d}\bar{x} \end{pmatrix} = \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & a_3 & 0 \\ 0 & 0 & a_4 \end{pmatrix} \begin{pmatrix} \mathrm{d}y - p\,\mathrm{d}x \\ \mathrm{d}p - f(x, y, p)\,\mathrm{d}x \\ \mathrm{d}x \end{pmatrix}.$$

For this problem, Cartan's equivalence method [Olv95] gives three fundamental invariants

$$\begin{cases} I_3 = -\dfrac{f_{ppp}a_4}{2{a_1}^2}, \\ I_2 = \dfrac{f_{yp} - D_x f_{pp}}{2a_1 a_4}, \\ I_1 = \dfrac{(2f_{yy} - D_x f_{yp} - f_{pp} f_y + f_{yp} f_p)a_1 + (-f_{yp} + D_x f_{pp})a_4 a_2}{2{a_1}^2 {a_4}^2} \end{cases}$$

and six invariant derivations defined on certain manifold $\tilde{M}$, fibered over J^1, with local coordinates de (x, y, p, a_1, a_2, a_4). Here, $D_x = \partial_x + p\partial_y + f\partial_p$ is the Cartan vector field.

When specializing on the Painlevé equations, the two fundamental invariants I_2 and I_3 vanish. On this splitting branch, the application of the Jaccobi identity to the final structure

[1] Indeed, as remarked by Cartan [Car24], the above equation can always be regarded as the geodesics equation of a projective structure on a surface with local coordinates x and y and thus invariant under point transformations.

equations shows that among the six invariant derivations only two can produce new invariants. These two derivations are

$$\begin{cases} X_1 = \dfrac{1}{a_1}\partial_y - \dfrac{a_2 a_4}{{a_1}^2}\partial_p - \dfrac{1}{2} f_{pp}\partial_{a_1} - \dfrac{1}{2}\dfrac{f_{py}}{a_4}\partial_{a_2}, \\ X_3 = \dfrac{1}{a_4}\partial_x + \dfrac{p}{a_4}\partial_y + \dfrac{f}{a_4}\partial_p + a_2\partial_{a_1} - \dfrac{f_y a_1}{{a_4}^2}\partial_{a_2} + \dfrac{2a_2 a_4 + f_p a_1}{a_1}\partial_{a_4}. \end{cases}$$

Notation 1. In the following, $I_{1;j\cdots k}$ denotes the differential invariant $X_k \cdots X_j(I_1)$. For instance, the invariant $I_{1;33}$ is obtained by differentiating twice the fundamental invariant I_1 with respect to invariant derivation X_3.

3. The first Painlevé equation $y'' = 6y^2 + x$

Since the associated fiber-preserving symmetry Lie pseudo-group is zero dimensional, this justifies the following lemma:

Lemma 1. *The specialization of the invariants*

$$I_1, I_{1;3}, I_{1;33}, \frac{I_{1;333}}{I_{1;33}}, \frac{I_{1;3333}}{I_{1;33}} - \frac{43}{120} I_{1;33}, \frac{I_{1;33333}}{I_{1;33}} - \frac{5}{4} I_{1;33}$$

on the first Painlevé equation gives six invariants functionally independent defined on $\tilde{M}$.

The problem with the above invariants is that they do depend on extra parameters a_1, a_2 and a_4. Fortunately, in our zero-dimensional case, we can normalize (i.e., eliminate) these parameters by setting

$$I_1 = -12, \qquad I_{1;3} = 0, \qquad \frac{I_{1;333}}{I_{1;33}} = 1. \tag{3.1}$$

Now substituting the values of the parameters in the remaining invariants gives us, due again to our zero-dimensional case, three functionally independent invariants that do not depend on the extra parameters. Writing the equality of the invariants and simplifying the obtained system, by computing a characteristic set [Kol73, BLOP95], give an algebraic transformation of degree 5:

$$\begin{cases} \bar{p} = 129600 \dfrac{(5{I_{1;33}}^2 + 4I_{1;33333})}{{I_{1;33}}^3} \bar{y}^4 \\ \bar{x} = -6 \dfrac{(120 I_{1;3333} + 43{I_{1;33}}^2)}{I_{1;33}^2} \bar{y}^2, \\ \bar{y}^5 = -\dfrac{1}{23328000} \dfrac{{I_{1;33}}^5}{(5{I_{1;33}}^2 + 4I_{1;33333})^2}. \end{cases} \tag{3.2}$$

In these formulae the invariants are normalized using (3.1), that is, they do not depend on the extra parameters. According to (ii) of the introduction and (1.2), we have the following theorem.

Theorem 1. *A second-order differential equation* $\mathcal{E}_f$ *is equivalent to the first Painlevé equation by a fiber-preserving transformation if and only if this transformation is given by (3.2) and the normalization (3.1).*

Let us explain how theorem 1 can be used in practice. Consider the following equations

$$y'' = c\frac{y'^2}{y} + \frac{1}{y}(y^4 + x), \tag{3.3}$$

and

$$y'' = c\frac{y'^2}{y} + y(y^4 + x). \tag{3.4}$$

The question is to determine the values of the parameter c for which the above equations can be mapped to the first Painlevé equation (and compute the equivalence transformation when the equivalence holds).

First of all, the fact that the derived invariants $I_{1;1}$ vanish on the first Painlevé equation restricts the possible values of c to $\{-1, 3\}$ for the first equation and to $\{-3, 5\}$ for the second equation.

The second step is to specialize (3.2) on the given equation to obtain transformation candidates. In step 3, we have to check whether the pullback of the first Painlevé equation w.r.t. these candidates is exactly the considered equation.

In the case of equation (3.3), the specialization yields

$$\begin{cases} \bar{p} = 36\dfrac{\bar{y}^4 p}{y^7}, \\ \bar{x} = 6\dfrac{\bar{y}^2 x}{y^4}, \\ \bar{y}^5 = \dfrac{1}{108}y^{10} \end{cases} \tag{3.5}$$

for $c = -1$ and

$$\begin{cases} \bar{p} = -864\dfrac{\bar{y}^4 y^5(625x^5 - 2079)(-25yx^3 + 250px^4 + 21y^3)}{(50x^3 + 3y^2)^4}, \\ \bar{x} = 6\dfrac{(2500x^5 - 891)y^4\bar{y}^2}{(50x^3 + 3y^2)^2}, \\ \bar{y}^5 = -\dfrac{1}{31104}\dfrac{(50x^3 + 3y^2)^5}{y^{10}(625x^5 - 2079)^2} \end{cases} \tag{3.6}$$

for $c = 3$. The third step shows that the equivalence holds only for $c = -1$ and the equivalence transformation is (3.5). We can also deduce, according to (ii) in the introduction, that the cardinal of the fiber-preserving (point) symmetry group of equation (3.3) with $c = -1$ is equal to 10.

The same calculations show that equation (3.4) can not be mapped to the first Painlevé equation. In particular, we have a *division by zero error* in step 2 for $c = 5$. *Warning:* this error does not mean that the method failed. In fact it is part of the method and implies that no equivalence transformation exists.

Time estimates are given in the tables where P_1 refers to the first Painlevé equation.

4. The second Painlevé equation $y'' = 2y^3 + xy + \alpha$

Again, due to the zero dimensionality, there exist seven invariants defined on the manifold of local coordinates $(x, y, p, a_1, a_2, a_4, \alpha)$ such that when specialized, on Painlevé two, they give exactly seven functionally independent functions. For instance, one can take the invariants $I_1, I_{1;3}, I_{1;31}, I_{1;33}, I_{1;331}, I_{1;3331}$ and $I_{1;33311}$. We normalize a_1, a_2 and a_4 by setting

$$I_1 = -12, \qquad I_{1;3} = -12, \qquad I_{1;31} = 0, \tag{4.1}$$

Table 1. Time estimates (in seconds) for $y'' = c\frac{y'^2}{y} + \frac{1}{y}(y^4 + x)$.

	Computation of transformation candidates	Checking equivalence with P_1
$c = -1$	0.15	(yes) 0.04
$c = 3$	2.13	(no) 0.13

Table 2. Time estimates (in seconds) for $y'' = c\frac{y'^2}{y} + y(y^4 + x)$.

	Computation of transformation candidates	Checking equivalence with P_1
$c = -3$	0.35	(no) 0.03
$c = 5$	Division by zero error	(no) 0.00

and as in the previous section, we obtain

$$\begin{cases} \bar{p} = \dfrac{1}{6}\left(\dfrac{I_{1;33311}(I_{1;3331} + 4032)}{I_{1;33311}I_{1;33} - 3096576 - 4032I_{1;331}}\right)\bar{y}^2\bar{\alpha}, \\ \bar{x} = -\left(16 + \dfrac{1}{72}I_{1;331}\right)\bar{y}^2, \\ \bar{y}^3 = 48384\dfrac{\bar{\alpha}}{I_{1;33311}I_{1;33} - 3096576 - 4032I_{1;331}}, \\ \bar{\alpha}^2 = -\dfrac{1}{112I_{1;33311}(16257024 + 8064I_{1;3331} + I_{1;3331}{}^2)}(I_{1;33311}{}^2I_{1;33}{}^2 \\ \quad - 8064I_{1;33311}I_{1;33}I_{1;331} - 6193152I_{1;33311}I_{1;33} \\ \quad + 9588782923776 + 24970788864I_{1;331} + 16257024I_{1;331}{}^2). \end{cases} \tag{4.2}$$

when $\alpha \neq 0$ and

$$\begin{cases} \bar{p} = \dfrac{1}{290304}I_{1;33311}(4032 + I_{1;3331})\bar{y}^5, \\ \bar{x} = -\dfrac{1}{72}(1152 + I_{1;331})\bar{y}^2, \\ \bar{y}^6 = -20901888\dfrac{1}{I_{1;33311}(4032 + I_{1;3331})^2}, \end{cases} \tag{4.3}$$

when $\alpha = 0$. The comparison with the symmetry pseudo-groups (1.3) and (1.4) proves the following theorem.

Theorem 2. *A second-order differential equation can be mapped to the second Painlevé equation* $y'' = 2y^3 + yx + \alpha$ *by a fiber-preserving transformation if and only if this transformation is given by (4.2) if* $\alpha \neq 0$ *and by (4.3) otherwise with the normalization (4.1) in both cases.*

Let us remark that (4.3) can be obtained from (4.2) (as well as (1.4) from (1.3)) by eliminating the $\bar{\alpha}$ and taking into account the functional dependence between the invariants resulting from $\bar{\alpha} = 0$. Nevertheless, it is safer to separate the two cases ($\alpha \neq 0$ and $\alpha = 0$).

5. Equivalence under point transformation

The equivalence problem under the more general point transformations naturally arises since Painlevé equations belong to the class of equations of the form

$$y'' = A(x, y) + B(x, y)y' + C(x, y)y'^2 + D(x, y)y'^3$$

which is invariant under point transformations. In this case our starting Pfaffian system is

$$\varphi^* \begin{pmatrix} \mathrm{d}\bar{y} - \bar{p}\,\mathrm{d}\bar{x} \\ \mathrm{d}\bar{p} - \bar{f}(\bar{x}, \bar{y}, \bar{p})\,\mathrm{d}\bar{x} \\ \mathrm{d}\bar{x} \end{pmatrix} = \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & a_3 & 0 \\ a_4 & 0 & a_5 \end{pmatrix} \begin{pmatrix} \mathrm{d}y - p\,\mathrm{d}x \\ \mathrm{d}p - f(x, y, p)\,\mathrm{d}x \\ \mathrm{d}x \end{pmatrix}$$

for which we normalize a_3 and prolong to obtain involution and four fundamental invariants defined on an eight-dimensional manifold. For the above class, only two invariants are not identically zero:

$$K_1 = (6f_{yy} - 4D_x f_{yp} + D_x{}^2 f_{pp} - 3f_y f_{pp} + 4f_{yp} f_p - D_x f_{pp} f_p)/(a_1 a_5{}^2),$$
$$K_2 = (2f_y f_{ppp} a_5 + 4f_{yp} f_p a_4 - D_x f_{pp} f_p a_4 - 3f_y f_{pp} a_4 - a_5 f_{pp} f_{yp} + a_5 f_{pp} D_x f_{pp} + 6a_4 f_{yy} + a_4 D_x D_x f_{pp} - a_5 D_x f_{ppp} f_p - a_5 f_{ppp} D_x f_p - 4a_4 D_x f_{yp} - 2f_{yyp} a_5 + 2a_5 D_x f_{ypp} - a_5 D_x D_x f_{ppp})/(a_5{}^2 a_1{}^2).$$

As in the fiber-preserving case, only two invariant derivations X_1 and X_3 (one page long) are needed.

Theorem 3. *A second-order ordinary differential equation $y'' = f(x, y, y')$ is equivalent*

(i) to the first Painlevé equation $y'' = 6y^2 + x$ under a point transformation if and only if this transformation is given by

$$\begin{cases} \bar{p} = \dfrac{5}{1056} \dfrac{(2^{15}3^5 11^3 K_{1;33333} + K_{1;33313}{}^3)}{K_{1;33313}{}^2} \bar{y}^4, \\ \bar{x} = -6 \dfrac{(2^9 3^3 5 11^2 K_{1;3333} + 43 K_{1;33313}{}^2)}{K_{1;33313}{}^2} \bar{y}^2, \\ \bar{y}^5 = -\dfrac{88}{375} \dfrac{K_{1;33313}{}^5}{(2^{15}3^5 11^3 K_{1;33333} + K_{1;33313}{}^3)^2} \end{cases} \tag{5.1}$$

with the normalization

$$K_1 = -12, \qquad K_2 = 0, \qquad K_{1;1} = 0, \qquad K_{1;3} = 0, \qquad K_{1;33}/K_{1;333} = 720.$$

(ii) to the second Painlevé equation $y'' = 2y^3 + xy + \alpha$ under a point transformation if and only if this transformation is given by

$$\begin{cases} \bar{p} = -\dfrac{1}{18} \dfrac{K_{2;3}(15K_{2;3}K_{1;33} - 216000 + 4032K_{2;3} - 450K_{1;331} - 50K_{2;3}K_{1;333})}{25K_{2;3}K_{1;33} - 115200 + 1728K_{2;3} - 150K_{1;331}} \bar{y}^2\bar{\alpha}, \\ \bar{x} = \dfrac{1}{3600}(25K_{2;3}K_{1;33} + 336K_{2;3} - 57600 - 50K_{1;331})\bar{y}^2, \\ \bar{y}^3 = -1800 \dfrac{\bar{\alpha}}{25K_{2;3}K_{1;33} - 115200 + 1728K_{2;3} - 150K_{1;331}}, \\ \bar{\alpha}^2 = -108 \dfrac{(25K_{2;3}K_{1;33} - 115200 + 1728K_{2;3} - 150K_{1;331})^2}{K_{2;3}(15K_{2;3}K_{1;33} - 216000 + 4032K_{2;3} - 450K_{1;331} - 50K_{2;3}K_{1;333})^2} \end{cases} \tag{5.2}$$

when $\alpha \neq 0$ *and*

$$\begin{cases} \bar{p} = -\dfrac{1}{16200} K_{2;3}(576K_{2;3} + 25K_{2;3}K_{1;333} + 30K_{2;3}K_{1;33} - 64800)\bar{y}^5, \\ \bar{x} = \dfrac{1}{1080}(5K_{2;3}K_{1;33} - 5760 - 72K_{2;3})\bar{y}^2, \\ \bar{y}^6 = -\dfrac{87480000}{K_{2;3}(576K_{2;3} + 25K_{2;3}K_{1;333} + 30K_{2;3}K_{1;33} - 64800)^2} \end{cases} \tag{5.3}$$

when $\alpha = 0$*, with the normalization*

$$K_1 = -12, \qquad K_2 = 0, \qquad K_{1;1} = 0, \qquad K_{1;3} = 0, \qquad K_{2;3}/K_{1;31} = -5/24.$$

Example. Let us terminate with considering the equivalence of the two equations (3.3) and (3.4) with the second Painlevé equation under point transformations. Here, computations are done with arbitrary c.

Equation (3.3). The specialization of (5.2) on this equation yields (after 0.512 s) a transformation candidate depending on c and which is too long to include in this paper. The variable $\bar{x}$ does not depend on p in only two cases $c \in \{-1, 3\}$ and these two values return a *division by zero error* when computing the other components. The same thing happens with the specialization of (5.3) on (3.3). Thus, equation (3.3) can not be equivalent to the second Painlevé equation under point transformations.

Equation (3.4). The specialization of (5.2) on this equation gives the following transformation (in 1.11 s):

$$\begin{cases} \bar{p} = \dfrac{1}{36}\dfrac{(c+3)(c-2)^2 p}{(1+c)(c-5)y^{12}} \times (9y^3c + 66y^6p + \cdots + 27y^3)\bar{\alpha}\bar{y}^2, \\ \bar{x} = \dfrac{2}{3}\dfrac{(-27y^6 + 3y^2xc - 2c^2p^2 - 24y^6c + 3y^6c^2 + 5cp^2 - 18y^2x + 6p^2 - c^3p^2 + 3y^2xc^2)}{(c-5)y^6} \\ \qquad \times \bar{y}^2, \\ \bar{y}^3 = \dfrac{1}{16}\dfrac{(c-5)}{1+c}\bar{\alpha}, \\ \bar{\alpha}^2 = 1728\dfrac{(5-c)(1+c)^2}{(c+3)(c-2)^2}y^{18} \times (-9y^3c - 66y^6p + 54y^2px + 18y^6c^2p - 48y^6pc \\ \qquad + 18y^2xc^2p + 2c^3p^3 - 2p^3c^2 + 72y^2pxc - 34p^3c - 30p^3 - 27y^3)^{-2}. \end{cases}$$

For the particular values of c for which $\bar{x}$ does not depend on p we obtain *division by zero errors* when computing the other components and then equation (3.4) can not be mapped to Painlevé two with $\alpha \neq 0$. However, the specialization of (5.3) on (3.4) gives

$$\begin{cases} \bar{p} = -\dfrac{4}{9}\dfrac{(c-2)^2p(c+3)(18xy^2pc^2 - 90py^6 + \cdots + 18y^6pc^2)}{(c-5)^2y^{12}}\bar{y}^5, \\ \bar{x} = \dfrac{2}{3}\dfrac{(-23y^6 + 3y^2xc - 2c^2p^2 - 20y^6c + 3y^6c^2 + 5cp^2 - 18y^2x + 6p^2 - c^3p^2 + 3y^2xc^2)}{(c-5)y^6} \\ \qquad \times \bar{y}^2, \\ \bar{y}^6 = -\dfrac{27}{4}\dfrac{(c-5)^3}{(c+3)(c-2)^2}y^{18} \times (18xy^2pc^2 - 90py^6 - 9y^3c + 54pxy^2 - 27y^3 + 72pcxy^2 \\ \qquad - 34p^3c - 72pcy^6 + 2c^3p^3 - 30p^3 - 2p^3c^2 + 18y^6pc^2)^{-2} \end{cases}$$

which is point transformation only when $c = -1$. In this case, the resulting transformation is

$$\bar{p} = 4\frac{\bar{y}^5 p}{y^9}, \qquad \bar{x} = 2\frac{\bar{y}^2 x}{y^4}, \qquad \bar{y}^6 = 1/4y^{12}$$

and this maps equation (3.4) to Painlevé two $y'' = 2y^3 + xy$ (with $\alpha = 0$).

Acknowledgments

The author wishes to thank Michel Petitot for fruitful discussions, Samuel Vidal for encouragement, Pavel Exner for financial support and anonymous referees for their valuable comments.

References

[BLOP95] Boulier F, Lazard D, Ollivier F and Petitot M 1995 Representation for the radical of a finitely generated differential ideal *Proc. ISSAC'95 (Montréal, Canada)* pp 158–66

[Car24] Cartan E 1924 Sur les variétés à connexion projective *Bull. Soc. Math. France* **52** 205–41

[CM08] Conte R M and Musette M 2008 *The Painlevé Handbook* (Dordrecht: Springer)

[DP07] Dridi R and Petitot M 2007 Towards a new ODE solver based on Cartan's equivalence method *ISSAC '07: Proc. Int. Symp. 2007 on Symbolic and Algebraic Computation (New York, NY, USA)* (New York: ACM) pp 135–42

[HD02] Hietarinta J and Dryuma V 2002 Is my ODE a Painlevé equation in disguise? *J. Nonlin. Math. Phys.* **9** 467–74

[KLS85] Kamran N, Lamb K G and Shadwick W F 1985 The local equivalence problem for $\mathrm{d}^2y/\mathrm{d}x^2 = F(x, y, \mathrm{d}y/\mathrm{d}x)$ and the Painlevé transcendents *J. Diff. Geom.* **22** 139–50

[Kol73] Kolchin E R 1973 *Differential Algebra and Algebraic Groups* (London: Academic)

[Olv93] Olver P J 1993 *Applications of Lie Groups to Differential Equations* (*Graduate Texts in Mathematics*) (Berlin: Springer)

[Olv95] Olver P J 1995 *Equivalence, Invariants, and Symmetry* (Cambridge: Cambridge University Press)